

A computer based classification of caps in $PG(4, 2)$

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Abstract

In this paper we present the complete classification of caps in $PG(4, 2)$. These results have been obtained using a computer based exhaustive search that exploits projective equivalence.

1 Introduction

In the projective space $PG(r, q)$ over the Galois Field $GF(q)$, a n -cap is a set of n points no 3 of which are collinear. A n -cap is called *complete* if it is not contained in a $(n + 1)$ -cap. For a detailed description of the most important properties of these geometric structures, we refer the reader to [4]. In the last decades the problem of determining the spectrum of the sizes of complete caps has been the subject of a lot of researches. For a survey see [1]. In [5] (see Th. 18.2.1) is presented the classification of complete caps in $PG(3, 2)$ and it is also possible to understand the classification of the incomplete caps. The following table shows the number of non equivalent complete and incomplete caps in $PG(3, 2)$.

Table 1: Number and type of non equivalent examples

$ \mathcal{K} $	# COMPLETE CAPS	# INCOMPLETE CAPS
5	1	1
6	0	1
7	0	1
8	1	0

In this work we search for the classification of complete and incomplete caps in $PG(4, 2)$, using an exhaustive search algorithm. In Section 2 the algorithm utilized is illustrated; in Section 3 the complete list of non equivalent complete and incomplete caps is presented.

2 The searching algorithm

In this section the algorithm utilized is presented. Our goal is to obtain the classification of complete and incomplete caps in $PG(4, 2)$. It is not restrictive to suppose that a cap in $PG(4, 2)$ contains this five points:

$$\mathcal{R} = \{(1 : 0 : 0 : 0 : 0); (0 : 1 : 0 : 0 : 0); (0 : 0 : 1 : 0 : 0); (0 : 0 : 0 : 1 : 0); (0 : 0 : 0 : 0 : 1)\}.$$

Then we define the set *Cand* of all the points lying no 2-secant of \mathcal{R} . We introduce in *Cand* the following equivalence relationship:

$$P \sim Q \iff \mathcal{C} \cup \{P\} \cong \mathcal{C} \cup \{Q\},$$

where \cong means that the two sets are projectively equivalent. This relationship spreads the candidates in equivalent classes $\mathcal{C}_1, \dots, \mathcal{C}_k$.

The choice of the next point to add to the building cap can be made only among the representatives of the equivalent classes, in fact two caps one containing $\mathcal{C} \cup \{P\}$ and the other one $\mathcal{C} \cup \{Q\}$, with P and Q in $\mathcal{C}_{\bar{i}}$, are equivalent by definition of orbit.

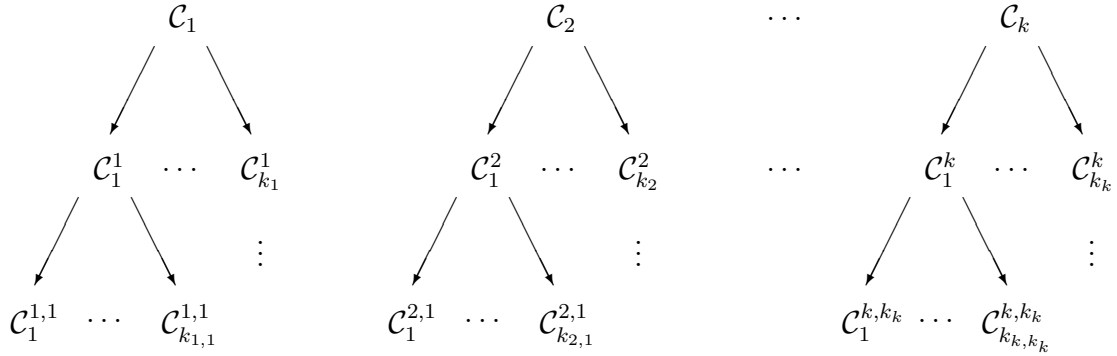
Suppose now that we have construct all the caps containing $\mathcal{C} \cup \{P_i\}$, with $i \leq \bar{i}$. Considering the caps containing $\mathcal{C} \cup \{P_j\}$ with $\bar{i} < j$, all the points of the classes \mathcal{C}_k with $k < \bar{i}$ can be avoided. In fact a cap containing $\mathcal{C} \cup \{P_j\} \cup \{\bar{P}_k\}$, with $\bar{P}_k \in \mathcal{C}_k$ and $k < \bar{i}$, is projectively equivalent to a cap containing $\mathcal{C} \cup \{P_k\} \cup \{P_j\}$, already studied.

When we add a new point to the cap, we can divide all the remaining candidates in equivalence classes, as above. Two points P and Q are in relationship with the j -th class \mathcal{C}_j , i.e.

$P \sim_j Q$, if $\mathcal{C} \cup \{P_j\} \cup \{P\}$ and $\mathcal{C} \cup \{P_j\} \cup \{Q\}$ are projectively equivalent.

At the m -th step of the extension process if the cap $\mathcal{C} \cup \{P_{i_m}\} \cup \dots \cup \{P_{i_1 \dots i_{m-1}}\} \cup \{P\}$ is projectively equivalent to the cap $\mathcal{C} \cup \{P_{i_m}\} \cup \dots \cup \{P_{i_1 \dots i_{m-1}}\} \cup \{Q\}$ with $P_s^{i_1 \dots i_r} \in \mathcal{C}_s^{i_1 \dots i_r}$, then P and Q are in relationship ($P \sim_{i_1 \dots i_m} Q$) and they belong to the same class $\mathcal{C}_{m+1}^{i_1 \dots i_m}$.

Iterating the process we can build a tree similar to the following:



The tree is important to restrict the number of candidates in the extension process. Suppose that we have generated a n -cap containing the cap $\mathcal{C} \cup \{P_{i_1}\} \cup \{P_{i_2}^{i_1}\} \cup \dots \cup \{P_{i_m}^{i_1 \dots i_{m-1}}\} \cup \{P\}$, after having generated n -caps containing $\mathcal{C} \cup \{P_j\}$ with $j < i_1$, $\mathcal{C} \cup \{P_{i_1}\} \cup \{P_j^{i_1}\}$ with $j < i_2, \dots$, $\mathcal{C} \cup \{P_{i_1}\} \cup \{P_{i_2}^{i_1}\} \cup \dots \cup \{P_j^{i_1 \dots i_{m-1}}\}$ with $j < i_m$, with $P_s^{i_1 \dots i_r} \in \mathcal{C}_s^{i_1 \dots i_r}$. Then the points belonging to $\mathcal{C}_1 \cup \dots \cup \mathcal{C}_{i_1-1} \cup \mathcal{C}_1^{i_1} \cup \dots \cup \mathcal{C}_{i_2-1}^{i_1} \cup \dots \cup \mathcal{C}_1^{i_1 \dots i_{m-1}} \cup \dots \cup \mathcal{C}_{i_m-1}^{i_1 \dots i_{m-1}}$ can be avoided, because a cap containing one of them is equivalent to one already found. For example a n -cap containing $\mathcal{C} \cup \{P_{i_1}\} \cup \dots \cup \{P_{i_m}^{i_1 \dots i_{m-1}}\} \cup \{P\} \cup \{Q\}$ with $Q \in \mathcal{C}_h$ for some $h < i_1$ is equivalent to a n -cap containing $\mathcal{C} \cup \{P_h\}$, which is already found.

3 Results

In this Section all non equivalent caps, complete and incomplete, in $PG(4, 2)$ are presented.

3.1 Non-equivalent caps \mathcal{K} in $PG(4, 2)$

This table shows the number and the type of the non equivalent examples of all the caps.

Table 2: Number and type of non equivalent examples

$ \mathcal{K} $	# COMPLETE CAPS	# INCOMPLETE CAPS
6	0	3
7	0	3
8	0	4
9	1	4
10	1	3
11	0	2
12	0	2
13	0	1
14	0	1
15	0	1
16	1	0

All the non equivalent examples of the caps described in the table above are presented. In particular for each cap we compute the stabilizer and the weight enumerators of the associated code.

Size	Type	Cap	Weight enumerators	G
6	Inc.	$\begin{array}{c} 1\ 0\ 0\ 0\ 0\ 1 \\ 0\ 1\ 0\ 0\ 0\ 1 \\ 0\ 0\ 1\ 0\ 0\ 1 \\ 0\ 0\ 0\ 1\ 0\ 1 \\ 0\ 0\ 0\ 0\ 1\ 1 \end{array}$	$2^{15}4^{15}6^1$	G_{720}
6	Inc.	$\begin{array}{c} 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 1\ 0\ 0\ 0\ 0 \\ 0\ 0\ 1\ 0\ 1\ 0 \\ 0\ 0\ 0\ 1\ 1\ 0 \\ 0\ 0\ 0\ 0\ 1\ 1 \end{array}$	$1^12^{10}3^{10}4^55^5$	G_{120}
6	Inc.	$\begin{array}{c} 1\ 0\ 0\ 0\ 0\ 1 \\ 0\ 1\ 0\ 0\ 0\ 0 \\ 0\ 0\ 1\ 0\ 0\ 1 \\ 0\ 0\ 0\ 1\ 0\ 1 \\ 0\ 0\ 0\ 0\ 1\ 0 \end{array}$	$1^22^73^{12}4^75^26^1$	G_{48}
7	Inc.	$\begin{array}{c} 1\ 0\ 0\ 0\ 0\ 1\ 0 \\ 0\ 1\ 0\ 0\ 0\ 0\ 1 \\ 0\ 0\ 1\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 1\ 0\ 1\ 1 \\ 0\ 0\ 0\ 0\ 1\ 0\ 1 \end{array}$	$2^63^94^95^67^1$	G_{72}
7	Inc.	$\begin{array}{c} 1\ 0\ 0\ 0\ 0\ 1\ 1 \\ 0\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 1\ 0\ 0\ 0\ 1 \\ 0\ 0\ 0\ 1\ 0\ 1\ 1 \\ 0\ 0\ 0\ 0\ 1\ 1\ 0 \end{array}$	$2^53^{12}4^75^46^3$	G_{48}
7	Inc.	$\begin{array}{c} 1\ 1\ 0\ 0\ 0\ 0\ 1 \\ 0\ 1\ 1\ 0\ 0\ 0\ 0 \\ 0\ 1\ 0\ 1\ 0\ 0\ 1 \\ 0\ 0\ 0\ 0\ 1\ 0\ 1 \\ 0\ 0\ 0\ 0\ 0\ 1\ 0 \end{array}$	$1^12^33^{11}4^{11}5^36^17^1$	G_{48}
8	Inc.	$\begin{array}{c} 1\ 1\ 1\ 0\ 0\ 0\ 0\ 1 \\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 0 \\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1 \\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1 \\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0 \end{array}$	$2^44^{22}6^48^1$	G_{384}
8	Inc.	$\begin{array}{c} 1\ 1\ 0\ 0\ 0\ 0\ 0\ 1 \\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0 \\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 1 \\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 1 \\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0 \end{array}$	$1^13^74^{14}5^77^18^1$	G_{168}
8	Inc.	$\begin{array}{c} 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1 \\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 1 \\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 1 \\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 0 \end{array}$	$2^13^{10}4^{11}5^46^37^2$	G_{48}
8	Inc.	$\begin{array}{c} 1\ 1\ 0\ 0\ 0\ 0\ 1\ 0 \\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 1 \\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 1 \\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1 \end{array}$	$2^23^84^{10}5^86^28^1$	G_{32}

Size	Type	Cap	Weight enumerators	G
9	Inc.	$\begin{matrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{matrix}$	$1^1 4^{14} 5^{14} 8^1 9^1$	G_{1334}
9	Inc.	$\begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{matrix}$	$3^4 4^{14} 5^8 7^4 8^1$	G_{192}
9	Inc.	$\begin{matrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{matrix}$	$3^6 4^9 5^9 6^6 9^1$	G_{72}
9	Inc.	$\begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{matrix}$	$2^1 3^3 4^{11} 5^{11} 6^3 7^1 9^1$	G_{48}
9	Comp.	$\begin{matrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{matrix}$	$2^1 4^{21} 6^7 8^2$	G_{336}
10	Inc.	$\begin{matrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{matrix}$	$4^{15} 6^{15} 10^1$	G_{720}
10	Inc.	$\begin{matrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{matrix}$	$2^1 4^6 5^{16} 6^6 8^1 10^1$	G_{384}
10	Inc.	$\begin{matrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{matrix}$	$3^2 4^7 5^{12} 6^7 7^2 10^1$	G_{48}
10	Comp.	$\begin{matrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{matrix}$	$4^{10} 5^{16} 8^5$	G_{1920}
11	Inc.	$\begin{matrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{matrix}$	$3^1 4^2 5^{12} 6^{12} 7^2 8^1 11^1$	G_{192}
11	Inc.	$\begin{matrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{matrix}$	$4^5 5^{10} 6^{10} 7^5 11^1$	G_{120}

Size	Type	Cap	Weight enumerators	G
12	Inc.	1 0 0 0 0 0 1 1 0 1 1 0 1 0 1 0 0 0 0 0 0 1 1 1 1 1 0 0 1 0 0 1 1 0 0 1 1 1 0 0 0 1 0 0 1 1 1 0 1 1 0 0 0 0 1 1 0 1 0 0 0 1	$4^3 6^{24} 8^3 12^1$	G_{2304}
12	Inc.	1 1 0 0 0 0 0 1 0 1 0 0 1 0 1 1 0 0 0 0 0 1 1 1 0 1 0 0 0 1 0 0 1 0 1 1 1 1 0 0 0 0 1 0 1 1 0 1 1 1 0 1 0 0 0 1 0 1 0 0 1 1	$4^1 5^8 6^{12} 7^8 8^1 12^1$	G_{192}
13	Inc.	1 0 0 0 0 0 1 1 0 1 1 0 0 1 0 1 0 0 0 0 0 0 1 1 1 1 0 1 0 0 1 0 0 1 1 0 0 1 1 1 1 0 0 0 1 0 0 1 1 1 0 1 1 1 0 0 0 0 1 1 0 1 0 0 0 1 1	$5^3 6^{12} 7^{12} 8^3 13^1$	G_{576}
14	Inc.	1 1 0 0 0 0 0 1 1 0 1 1 0 0 1 0 1 1 0 0 0 0 0 0 1 1 1 1 0 1 0 0 0 1 0 0 1 1 0 0 1 1 1 1 0 0 0 0 1 0 0 1 1 1 0 1 1 1 0 1 0 0 0 1 1 0 1 0 0 0 1 1	$6^7 7^{16} 8^7 14^1$	G_{2688}
15	Inc.	1 1 0 0 0 0 0 1 1 1 0 1 1 0 0 1 0 1 1 0 0 0 0 0 0 0 1 1 1 1 0 1 0 0 0 1 0 0 1 1 0 0 0 1 1 1 1 0 0 0 0 1 0 0 1 1 1 1 0 1 1 1 0 1 0 0 0 1 1 0 1 1 0 0 0 1 1	$7^{15} 8^{15} 15^1$	G_{20160}
16	Comp.	1 1 0 0 0 0 0 1 1 1 0 1 1 0 0 0 1 0 1 1 0 0 0 0 0 0 0 1 1 1 1 1 0 1 0 0 0 1 0 0 1 1 0 0 0 1 1 1 1 1 0 0 0 0 1 0 0 1 1 1 1 0 0 1 1 1 0 1 0 0 0 1 1 0 1 1 0 0 1 0 1 1	$8^{30} 16^1$	G_{322560}

References

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